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Chiral Potts Model as a Descendant of the Six-Vertex Model

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We observe that the N-state integrable chiral Potts model can be considered as a part of some new algebraic structure related to the six-vertex model. As a result, we obtain a functional equation which is supposed to determine all the eigenvalues of the chiral Potts model transfer matrix.

KEY WORDS: Yang–Baxter equation; Hopf algebra; chiral Potts model; functional equations; transfer matrix; high-genus curves.

1. INTRODUCTION

The star–triangle (or Yang–Baxter) relation and its generalization play a central role in the theory of exactly solvable models in statistical mechanics[(1,2)] and field theory.[(3)] Many solutions of the Yang–Baxter equations have been found. Usually they are uniformized in terms of elementary functions or Jacobi elliptic functions of fixed modulus, where the argument is the difference of the "rapidities" of the two lines through that vertex.

Recently, solutions of the star–triangle equations have been found[(4–6)] that were shown not to be of this form. In fact, they should be uniformized by genus $g > 1$ curves and hence cannot have the difference property. The general solution of this type has been found in ref. 7 for the N-state chiral Potts models. Note that some of the Hamiltonians associated with these models were studied early in refs. 8–10.

In the present paper we show that, in spite of all its unusual properties, the chiral Potts model has a very close relation to the "conventional" integrable models. In fact, we show that the integrable

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chiral Potts model can be considered as part of a new algebraic structure related to the six-vertex model.[11,12] Note that an explicit manifestation of such a connection for the spectrum of the superintegrable 3-state chiral Potts model was observed in refs. 13 and 14.

We find some new 2 by N L-operators as solutions of the Yang–Baxter equation with the six-vertex model R-matrix. These L-operators satisfy another Yang–Baxter equation with an N-state R-matrix. We show that this R-matrix is precisely that of the N-state chiral Potts model.

The organization of this paper is as follows. The L-operators which are interwounded by the six-vertex R-matrix are obtained in Section 2. Section 3 contains basic definitions of the chiral Potts model. In Section 4 we establish a relationship between six-vertex and chiral Potts models. In Section 5 we derive various functional relations for the transfer matrix of the chiral Potts model. We suppose that these relations should determine all the eigenvalues of this transfer matrix. For the $N=3$ case our relations imply the one presented in ref. 15.

Baxter observed various interconnections of our results with his recent results on superintegrable chiral Potts model.[16-18] These connections are discussed in detail in ref. 19 (see also the end of Section 4 of the present paper).

2. L-OPERATORS RELATED TO THE SIX-VERTEX MODEL R-MATRIX

In this section we consider some solutions of the Yang–Baxter equations (YBE) related to the six-vertex model R-matrix. The latter is a four-index matrix function $R_{i_1 i_2}^{j_1 j_2}(x)$ (the indices run over the two values 0 and 1) with the following nonvanishing matrix elements:

$$\begin{aligned} &R_{00}^{00}=R_{11}^{11}=\rho\sin(\theta+\eta), &&R_{01}^{01}=R_{10}^{10}=\rho\sin\theta \\ &R_{01}^{10}=\rho\sin\eta e^{i\theta}, &&R_{10}^{01}=\rho\sin\eta e^{-i\theta} \end{aligned} \tag{2.1}$$

where $\theta=-i\log x$ is a variable, while ρ, η are considered as constants. $R(x)$ satisfies the YBE

$$R_{i_3 i_1}^{j_3 j_1}(x) R_{j_3 i_2}^{k_3 j_2}(y) R_{j_1 j_2}^{k_1 k_2}(yx^{-1}) = R_{i_1 i_2}^{j_1 j_2}(yx^{-1}) R_{i_3 j_2}^{j_3 k_2}(y) R_{j_3 j_1}^{k_3 k_1}(x) \tag{2.2}$$

where summation over repeated indices is assumed (see Fig. 1).

Let $L(x)$ be an operator in $C^2 \otimes C^N$, $N \geqslant 2$, satisfying the following equation (shown in Fig. 2):

$$L_{i_1 \alpha}^{j_1 \beta}(x) L_{i_2 \beta}^{j_2 \gamma}(y) R_{j_1 j_2}^{k_1 k_2}(yx^{-1}) = R_{i_1 i_2}^{j_1 j_2}(yx^{-1}) L_{j_2 \alpha}^{k_2 \beta}(y) L_{j_1 \beta}^{k_1 \gamma}(x) \tag{2.3}$$

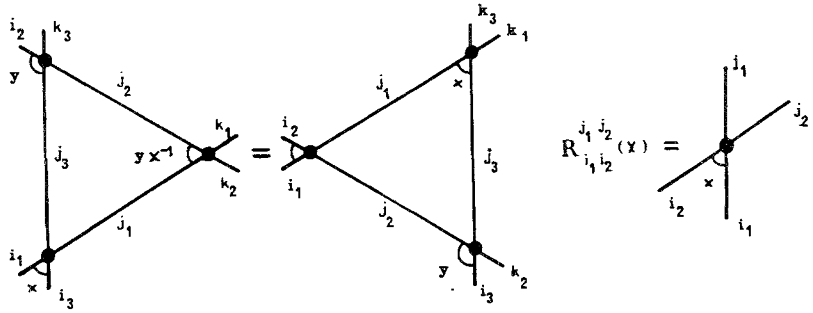

Fig. 1. Graphical representation of the Yang–Baxter equation (2.2).

where $L^{j\beta}_{i\alpha}(x)$ $(i, j = 0, 1;\ \alpha, \beta = 0, 1,..., N-1)$ denote the matrix elements of $L(x)$. The operator $L(x)$ is called a quantum L-operator related to a given R-matrix. It can conveniently be viewed as a two by two matrix with operator matrix elements acting in C^N. Then, one can rewrite (2.3) as

$$[L(x) \otimes L(y)]\,\check{R}(yx^{-1}) = \check{R}(yx^{-1})[L(y) \otimes L(x)] \tag{2.3'}$$

where

$$\check{R}(x) = R(x)P \tag{2.4}$$

P is a permutation operator in $C^2 \otimes C^2$, $P(x \otimes y) = (y \otimes x)$.

Discarding an interesting question about the most general solution of (2.3), let us search for an L-operator of the form

$$L(x) = xL_+ + x^{-1}L_- \tag{2.5}$$

where L_+ (L_-) is independent of x and has an upper (lower) triangular form. The most obvious nontrivial solution of this form for $N = 2$ is the R-matrix itself. From (2.1), (2.4) it follows that

$$R(x) = \frac{\rho}{2}(xR_+ + x^{-1}R_-) \tag{2.6}$$

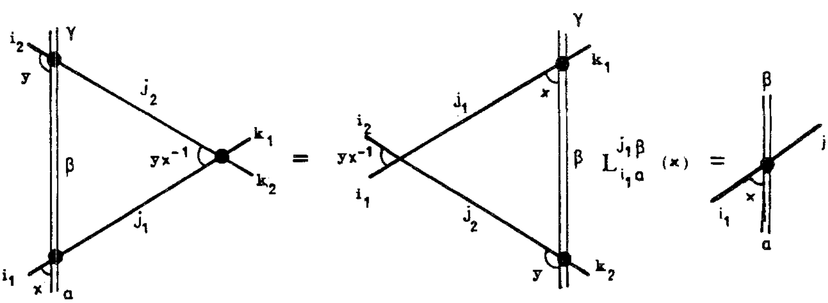

Fig. 2. Graphical representation of the Yang–Baxter equation (2.3).

where the $R_{\pm}$ are independent of x. Introducing $\check{R}_{\pm}=R_{\pm}P$ as in (2.4), we have

$$\begin{aligned} \check{R}_{+}\check{R}_{-}&=\mathbf{1} \\ \check{R}_{+}+\check{R}_{-}&=2\sin\eta\,\mathbf{1} \end{aligned} \tag{2.7}$$

By using (2.5)–(2.7), we find that Eq. (2.3) reduces to only three independent relations,

$$\begin{aligned} (L_{\pm}\otimes L_{\pm})\check{R}_{+}&=\check{R}_{+}(L_{\pm}\otimes L_{\pm}) \\ (L_{-}\otimes L_{+})\check{R}_{+}&=\check{R}_{+}(L_{+}\otimes L_{-}) \end{aligned} \tag{2.8}$$

Explicitly, we have

$$\begin{gathered} [(L_{\sigma})_{ii},(L_{\sigma'})_{jj}]=0, \qquad \sigma,\sigma'=\pm, \quad i,j=0,1 \\ (L_{\sigma})_{ii}(L_{+})_{01}=\omega_1^{-\sigma\varepsilon(i)}(L_{+})_{01}(L_{\sigma})_{ii}, \qquad \sigma=\pm, \quad i=0,1 \\ (L_{\sigma})_{ii}(L_{-})_{10}=\omega_1^{\sigma\varepsilon(i)}(L_{-})_{10}(L_{\sigma})_{ii}, \qquad \sigma=\pm, \quad i=0,1 \\ [(L_{+})_{01},(L_{-})_{10}]=(\omega_1-\omega_1^{-1})\{(L_{-})_{11}(L_{+})_{00}-(L_{+})_{11}(L_{-})_{00}\} \end{gathered} \tag{2.8'}$$

where $\varepsilon(0)=1$, $\varepsilon(1)=-1$, and $\omega_1=\exp(i\eta)$. These relations can be considered as a definition of some quadratic Hopf algebra[20,21] with six generating elements $(L_{\sigma})_{ii}$, $i=0,1$, $\sigma=\pm$, and $(L_{+})_{01}$, $(L_{-})_{10}$, which generalizes the $U_q(sl(2))$ algebra.[22] The latter arises if we set, e.g.,

$$(L_{\sigma})_{00}=(L_{-\sigma})_{11}, \qquad \sigma=\pm \tag{2.9}$$

We are interested in representations of the algebra (2.8) which, in general, do not match the above constraints. Moreover, let us require that

$$\det_{C^N}(L_{\sigma})_{ij}\neq 0 \tag{2.10}$$

for all values of σ, i, j. From the relations (2.8) it follows that this property can be achieved only if $\omega_1^N=1$, i.e.,

$$\eta=2\pi k/n, \qquad k=0,1,\ldots,N-1 \tag{2.11}$$

Here we restrict ourselves to the case when

$$N=\text{prime number} \tag{2.11'}$$

Then one can show that the most general[2] solution of (2.8), (2.10), and (2.11) can be written as

[2] When N is not prime, Eqs. (2.12) and (2.13) still give a solutions of (2.8) and (2.10), but apparently not the most general one.

$$\begin{aligned} D_+ &= (L_+)_{00} = d_+ A \\ D_- &= (L_-)_{00} = d_- B \\ F_+ &= (L_+)_{11} = f_+ B \\ F_- &= (L_-)_{11} = f_- A \\ G &= (L_+)_{01} = (g_+ B + g_- A) C \\ H &= (L_-)_{10} = (h_+ B + h_- A) C^{-1} \end{aligned} \tag{2.12}$$

where A, B, and C are N by N matrices satisfying the relations

$$[A, B] = 0; \qquad CA = \omega_1 AC; \qquad CB = \omega_1^{-1} BC \tag{2.13}$$

The eight parameters d_+, d_-, f_+, f_-, g_+, g_-, h_+, and h_- are arbitrary *modulo* the constraints

$$g_- h_- = f_- d_+, \qquad g_+ h_+ = f_+ d_- \tag{2.14}$$

So we can choose a set of six parameters

$$\chi = \{d_+, d_-, f_+, f_-, g_+, g_-\} \tag{2.15}$$

as the independent ones.

Thus, Eqs. (2.5) and (2.12)–(2.14) define a six-parameter solution of the YBE (2.3) for the case (2.11). A particular choice of the matrices A, B, and C in (2.12) convenient for subsequent calculations is

$$A = X^{\rho}, \qquad B = X^{-\rho}, \qquad C = Z \tag{2.16}$$

where $\rho = (N-1)/2$ and X and Z are N by N matrices,

$$\begin{gathered} X_{\alpha\beta} = \delta_{\alpha, \beta+1}, \qquad Z_{\alpha\beta} = \delta_{\alpha\beta} \omega^{\alpha} \\ ZX = \omega XZ \end{gathered} \tag{2.17}$$

$$\delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta \pmod N \\ 0, & \alpha \neq \beta \pmod N \end{cases} \tag{2.18}$$

where $\omega = \omega_1^{-2}$ and $\alpha, \beta = 0, \ldots, N-1$.

Let us discuss some properties of the transfer matrices associated with $L(x)$ given by (2.5) and (2.12)–(2.14). For the lattice of M by M' sites the column-to-column and row-to-row transfer matrices have the form (see Figs. 3 and 4, respectively)

$$T_{\text{col}} = \mathscr{T}(x, \chi)_{i_1, \ldots, i_N}^{j_1, \ldots, j_N} = \sum_{\{\alpha\}} \prod_{k=1}^{M'} (L_{i_k \alpha_{k+1}}^{j_k \alpha_k}) \tag{2.19}$$

$$T_{\text{row}} = T(x, \chi)_{\alpha_1, \ldots, \alpha_N}^{\beta_1, \ldots, \beta_N} = \sum_{\{i\}} \prod_{k=1}^{M} (L_{i_k \;\; \alpha_k}^{i_{k+1} \beta_k}) \tag{2.20}$$

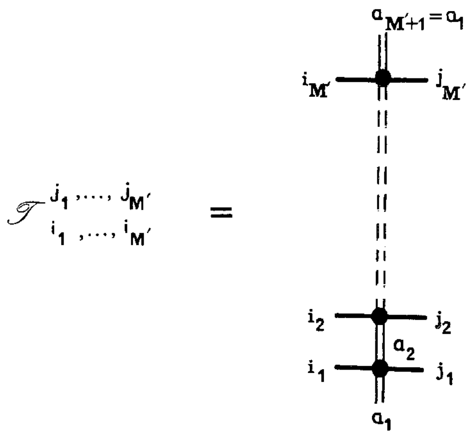

Fig. 3. Graphical representation of the column-to-column transfer matrix (2.19).

The first one acts in $(C^2 \otimes)^{M'}$, while the second one acts in $(C^N \otimes)^M$. In addition, we introduce a six-vertex model transfer matrix acting in $(C^2 \otimes)^{M'}$,

$$T_{6v}(x)^{j_1,\ldots,j_N}_{i_1,\ldots,i_N} = \sum_{\{k\}} \prod_{s=1}^{M'} R^{j_s k_{s+1}}_{i_s k_s}(x) \tag{2.21}$$

with R given by (2.1). It follows from (2.3) that

$$[\mathscr{T}(x, \chi), T_{6v}(y)] = 0 \tag{2.22}$$

It is well known that T_{6v} commutes with an arrow number operator $\mathscr{N}$,

$$[T_{6v}, \mathscr{N}] = 0, \qquad \mathscr{N} = \sum_{k=1}^{N} 1 \otimes \cdots \otimes \underset{k\text{th}}{\sigma_z} \otimes \cdots 1 \tag{2.23}$$

Contrary to this, the transfer matrix $\mathscr{T}(x, \chi)$ does not commute with $\mathscr{N}$. This intriguing phenomenon is possibly due to the degeneracy of the spectrum of T_{6v} among sectors with values of $\mathscr{N}$ differing by multiples of

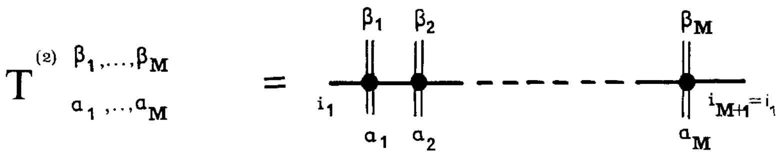

Fig. 4. Graphical representation of the row-to-row transfer matrix (2.20).

N. Thus, Eq. (2.22) implies the existence of a family of new arrow-nonpreserving integrals, commuting with the 6v-model transfer matrix (as we shall see in Section 4, these integrals, in general, do not commute among themselves).

Another interesting feature of $\mathscr{T}(x, \chi)$ is that it possesses the properties of Baxter's Q-matrix [Eq. (10.5.32) of ref. 2] for the 6v model. Namely, one can show that

$$T_{6v}(\theta)\,\mathscr{T}(\theta) = \sin^N \theta\,\mathscr{T}(\theta+\eta) + \sin^N(\theta+\eta)\,\mathscr{T}(\theta-\eta)$$
$$\mathscr{T}(\theta) = \mathscr{T}(e^{i\theta}, \chi(\theta))$$
$$\chi(\theta) = \{a, b, ce^{-i\theta}, de^{i\theta}, \lambda b, \lambda a\}$$

where $\theta = -i \log x$ and a, b, c, d, and λ are arbitrary parameters.

The properties of $T(x, \chi)$ are interesting as well. In particular, we shall show in Section 4 that it commutes with the transfer matrix of the integrable checkerboard N-state chiral Potts model.

3. CHIRAL POTTS MODEL

Following ref. 7, let us recall the basic definitions of the checkerboard integrable chiral Potts model. Consider an oriented square lattice $\mathscr{L}$ and its dual $\mathscr{L}'$ (shown in Fig. 5 by solid and dashed lines, respectively).

The vertical lines of $\mathscr{L}'$ carry rapidity variables q, q' in alternating order. Each rapidity variable q is represented by a 4-vector (a_q, b_q, c_q, d_q) restricted to lie on a curve. Similarly, the horizontal lines carry the rapidity variables p, p'. Place spin variables $\sigma = 0,..., n-1$ on sites of the original lattice $\mathscr{L}$. Then there are two kinds of neighboring spin pairs, as indicated in Fig. 6, with states a and b, and Boltzmann weights $W_{pq}(a-b)$ and $\bar{W}_{pq}(a-b)$ on the edges of $\mathscr{L}$. Here the arrow from a to b indicates that

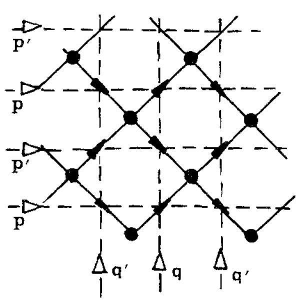

Fig. 5. Part of the lattice for the formulation of the checkerboard chiral Potts model. The open arrows show the directions of the rapidities.

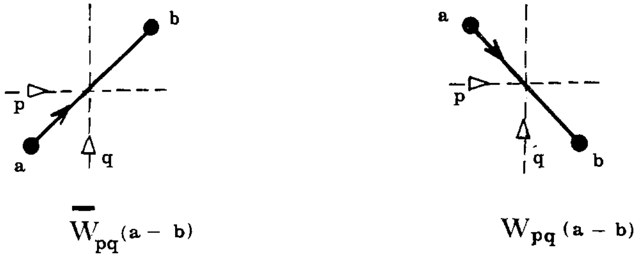

Fig. 6. The two types of Boltzmann weights.

the arguments is $a-b$ (mod N), rather than $b-a$. This arrow corresponds to the chirality of the model.

We can write down the star–triangle equation both graphically, as in Fig. 7, and algebraically, i.e.,

$$\sum_{e=1}^{N} \bar{W}_{qp}(a-e)\, W_{rp}(c-e)\, \bar{W}_{rq}(e-b) = R_{rqp}\, W_{rq}(c-a)\, \bar{W}_{rp}(a-c)\, W_{qp}(c-b) \tag{3.1}$$

where R_{rpq} is independent of the spins. The solution of (3.1) found in ref. 7 has the form

$$\frac{W_{pq}(k)}{W_{pq}(0)} = \prod_{j=1}^{k} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} \tag{3.2}$$

$$\frac{\bar{W}_{pq}(k)}{\bar{W}_{pq}(0)} = \prod_{j=1}^{k} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \tag{3.3}$$

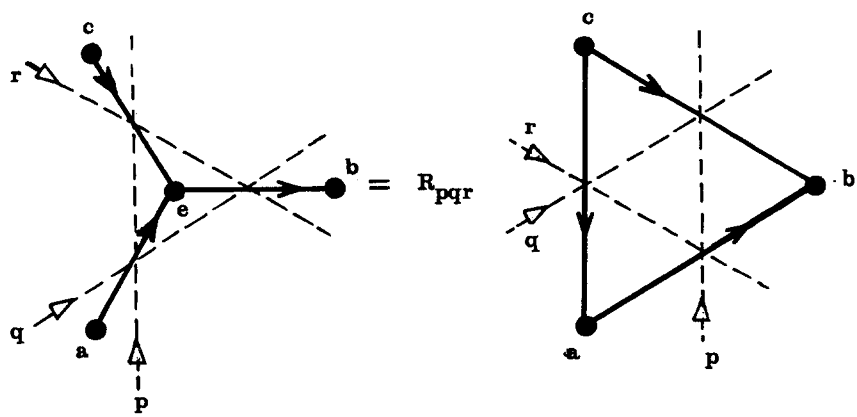

Fig. 7. Graphical representation of the star–triangle relation (3.1).

where

$$\omega = \exp(2\pi i/N) \tag{3.4}$$

Each rapidity 4-vector

$$x_p = (a_p, b_p, c_p, d_p)$$

associated with a line p is restricted to satisfy the following relations:

$$a_p^N + k' b_p^N = k d_p^N \tag{3.5a}$$

$$k' a_p^N + b_p^N = k c_p^N \tag{3.5b}$$

$$k a_p^N + k' c_p^N = d_p^N \tag{3.5c}$$

$$k b_p^N + k' d_p^N = c_p^N \tag{3.5d}$$

where $k'^2 = 1 - k^2$. Note that each pair of (3.5) implies the other two. The constant k is a parameter of the model. In the particular self-dual case $k = 0$ the model is reduced to the Fateev–Zamolodchikov model[23] (Z_N model). The latter model is critical.

Let $\sigma_1,..., \sigma_M$ and $\sigma'_1,..., \sigma'_M$ be the spins of two adjacent N-site rows of $\mathscr{L}$. Then, for cylindrical boundary conditions, one can define two transfer matrices,

$$(U_{p,q,q'})_{\sigma\sigma'} = \prod_{i=1}^{N} \bar{W}_{pq}(\sigma_i - \sigma'_i)\, W_{pq'}(\sigma'_i - \sigma_{i+1}) \tag{3.6a}$$

$$(\hat{U}_{p,q,q'})_{\sigma\sigma'} = \prod_{i=1}^{N} \bar{W}_{pq'}(\sigma_i - \sigma'_{i+1})\, W_{pq}(\sigma'_i - \sigma_i) \tag{3.6b}$$

Clearly,

$$\hat{U}_{p,q,q'} = U_{p,q',q} \hat{P} \tag{3.7}$$

where $\hat{P}$ shifts the spins one site

$$\hat{P}_{\sigma\sigma'} = \prod_{i=1} \delta_{\sigma_{i-1},\sigma_i} \tag{3.8}$$

For the homogeneous case when $q' = q$ the transfer matrices $U_{p,q,q}$, $\hat{U}_{p',q,q}$ with the same value of q but with different values of p commute among each other and with the operator P

$$[U_{pq}, \hat{U}_{p'q}] = [U_{pq}, P] = [\hat{U}_{pq}, P] = 0 \tag{3.9}$$

This is the consequence of star–triangle relation (3.1).

4. CHIRAL POTTS MODEL AS A DESCENDANT OF THE SIX-VERTEX MODEL

As was noted in Section 2, the transfer matrix (2.19) commutes with the 6-vertex model transfer matrix (2.21). Nevertheless two different transfer matrices (2.9) do not necessarily commute among themselves, because of the degeneracy of the spectrum of the 6-vertex model.

Let L and $\tilde{L}$ be two L-operators of the form (2.5), (2.12) with different sets of parameters χ, $\tilde{\chi}$, (2.15). It is convenient so set $x=1$ in (2.5) because it can be absorbed into the other parameters. Then we have

$$L=\begin{pmatrix} D & G \\ H & F \end{pmatrix} \tag{4.1}$$

where $D=D_+ + D_-$, $F=F_+ + F_-$, the other notations being defined by (2.12).

Clearly, the transfer matrices corresponding to L and $\tilde{L}$ will commute if there exists an intertwining matrix S satisfying the equation

$$\sum_{i_2=0}^{1} \sum_{\alpha_2,\beta_2=0}^{n-1} L_{i_1\alpha_1}^{i_2\alpha_2} \tilde{L}_{i_2\beta_1}^{i_3\beta_2} S_{\alpha_2\beta_2}^{\alpha_3\beta_3} = \sum_{i_2=0}^{1} \sum_{\alpha_2,\beta_2=0}^{n-1} S_{\alpha_1\beta_1}^{\alpha_2\beta_2} \tilde{L}_{i_1\beta_2}^{i_2\beta_3} L_{i_2\alpha_2}^{i_3\alpha_3} \tag{4.2}$$

or, using a matrix notation,

$$(L_i^j \otimes \tilde{L}_j^k)\, \check{S} = \check{S}(\tilde{L}_i^j \otimes L_j^k) \tag{4.3}$$

where $(L_i^j \otimes \tilde{L}_j^k)$ now denotes a matrix product in C^2 and a direct product in $C^N \otimes C^N$ (the summation over repeated indices is assumed).

The last equation implies (in the case that S exists) that any of the

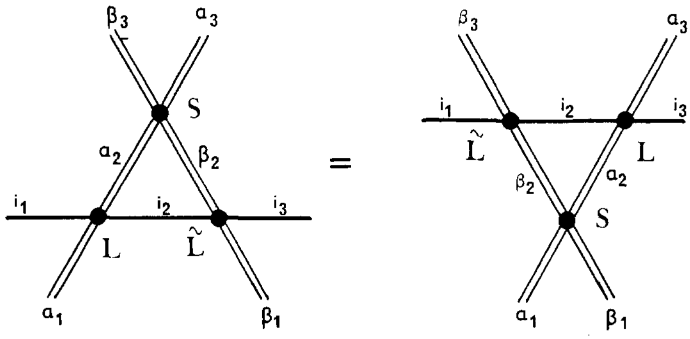

Fig. 8. Graphical representation of the Yang–Baxter equation (4.2).

four pairs of matrices $(L_i^j \otimes \tilde{L}_j^k)$ and $(\tilde{L}_i^j \otimes L_j^k)$, for $i, k = 0, 1$, should have the same spectra. Requiring this, one can show that the three quantities

$$\Gamma_1 = \frac{(d_+^N - f_+^N)(d_-^N - f_-^N)}{(h_+^N + h_-^N)(g_+^N + g_-^N)}, \qquad \Gamma_2 = \frac{d_-^N - f_-^N}{d_+^N - f_+^N} \tag{4.4}$$

$$\Gamma_3 = \frac{h_+^N + h_-^N}{g_+^N + g_-^N} \tag{4.5}$$

should be the same for L and $\tilde{L}$.

At this stage it is convenient to introduce a parametrization for the coefficients in (2.2). Obviously, the matrix $\check{S}$ in (4.3) is unaffected by a simultaneous similarity transformation of L and $\tilde{L}$ considered as matrices in C^2,

$$L_i^j \to A_{ik} L_k^l (A^{-1})_{lj}, \qquad \tilde{L}_i^j \to A_{ik} \tilde{L}_k^l (A^{-1})_{lj} \tag{4.6}$$

If we choose $A = \mathrm{diag}(\lambda^{1/2}, \lambda^{-1/2})$, then (4.6) results only in a rescaling of the parameters g_+, g_-, h_+, h_- in (2.12). The values of Γ_1, Γ_2 remain unchanged, while Γ_3 rescales as

$$\Gamma_3 \to \Gamma_3 \lambda^{-2N} \tag{4.7}$$

Now, choose the modulus k and three points p, q, q' on the curve (3.5) such that

$$\Gamma_1 = k^2, \qquad \Gamma_2 = \frac{a_p^N n_p^N}{d_p^N c_p^N} \tag{4.8}$$

Then, using the transformation (4.6), (4.7) one can adjust Γ_3 so that

$$\Gamma_3 = -\frac{a_p^N c_p^N}{d_p^N b_p^N} \tag{4.9}$$

Note that in this gauge we have the relation

$$\det F - \det D + \det H - \det G = 0 \tag{4.10}$$

where D, F, G, and H are defined by (4.1). The coefficients in (2.12) can be parametrized as

$$\begin{aligned} d_+ &= -\rho_1 c_p d_p b_q b_{q'}, & h_+ &= \omega \rho_1 a_p c_p d_q a_{q'} \\ d_- &= \omega \rho_1 a_p b_p d_q d_{q'}, & h_- &= -\omega \rho_1 a_p c_p b_q c_{q'} \\ f_+ &= -\omega \rho_1 c_p d_p a_q a_{q'}, & g_+ &= -\omega \rho_1 b_p d_p a_q d_{q'} \\ f_- &= \omega \rho_1 a_p b_p c_q c_{q'}, & g_- &= \rho_1 b_p d_p c_q b_{q'} \end{aligned} \tag{4.11}$$

where $\omega = \omega_1^{-2}$. Equations (4.11) contain five independent parameters ρ_1, k, p, q, q' instead of the six in (2.15). One parameter was absorbed by the gauge transformation (4.6), (4.7). Note that the transfer matrix (2.20) does not depend on this gauge degree of freedom. In particular, applying the transformation (4.6) with $\lambda = c_p/b_p$ to the coefficients (4.11), one can show that the transfer matrix (2.20) can be written as

$$T^{(2)}(p; q, q') = T(1, \chi) = (c_p d_p c_q c_{q'})^M P(t_p) \tag{4.12}$$

where $P(t_p)$ is an Mth-degree polynomial in the variable

$$t_p = a_p b_p / c_p d_p$$

Let us turn to the calculation of S in (4.3). Parametrize the coefficients of $\tilde{L}$ by the same formulas (4.11) with q, q' replaced by r, r', respectively. Solving now the linear system (4.2) for the elements S, we obtain the following unexpected result:

$$S_{\alpha\gamma}^{\beta\delta}(q, q', r, r') = W_{qr'}(\alpha, \gamma)\, \bar{W}_{q',r'}(\alpha, \beta)\, \bar{W}_{q,r}(\gamma, \delta)\, W_{q',r}(\beta, \delta) \tag{4.13}$$

where W and $\bar{W}$ are the Boltzmann weight of the chiral Potts defined by (3.2), (3.3). Note that the matrix (4.13) was used[7] for the vertex formulation (i.e., with spins placed on the edges of the lattice) of the checkerboard chiral Potts model. Using the star–triangle relation (3.1), one can show that it satisfies the Yang–Baxter equation.[7]

Recall now how we came to the chiral Potts model. We started from the six-vertex model R-matrix (2.1) satisfying the YBE (2.2). Then we solved another more general YBE (2.3) [with the Ansatz (2.5), (2.10), (2.11)], which includes this R-matrix as an "input." As a result we obtained the L-operators (2.12). Finally, the R-matrix S, (4.13), was found as a solution of the third YBE (4.2), which in turn includes as an input these L-operators. Thus, the chiral Potts model appeared here as the result of some unambiguous procedure, which exhibits some new algebraic structure related to the six-vertex model.

Before ending this section, we make two remarks about the structure of the L-operator (4.1), (2.12). Baxter observed that our $T^{(2)}(p; q, q')$ coincides with T_{col}^{-1} in his recent paper[18] on the superintegrable chiral Potts model [Eqs. (8.7), (8.13) of ref. 18]. In fact, using (2.12), (4.1), and (4.11), one can easily rewrite the matrix elements of L as

$$L_{i\alpha}^{j\beta} = f_i(\alpha - \beta)\, h_j(\alpha - \beta)$$

where $f_i(\alpha - \beta)$ depends only on p, q, k, while $g_j(\alpha - \beta)$ depends only on p, q', k. These connections are discussed in ref. 19.

Korepin and Tarasov[28] noticed that the L-operator (4.1) can be decomposed into a product of two more elementary L-operators of a massless lattice sine-Gordon model found in ref. [24]. Let X_1, Z_1 and X_2, Z_2 be the two sets of matrices (2.13) acting in different N-dimensional spaces. Then

$$L=\begin{pmatrix} aX_1 & bZ_1 \\ cZ_1^{-1} & dX_1^{-1} \end{pmatrix}\begin{pmatrix} \tilde{a}X_2 & \tilde{b}Z_2 \\ \tilde{c}Z_2^{-1} & \tilde{d}X_2^{-1} \end{pmatrix}$$

if we set $X_1X_2Z_2^{-1}Z_1=1$ and identify $X_2^{-1}Z_2$, $Z_2^{-1}X_2$, X_1X_2 with A, B, C in (2.12), respectively.

5. FUNCTIONAL RELATIONS

We wish to find the eigenvalues of the matrix $\mathbf{T}$ defined by (2.20). Just as in ref. 25, let us first search for vectors whose elements are products of single spin functions,

$$\mathbf{Q}_{\{\alpha\}}=\phi^1_{\alpha_1}\phi^2_{\alpha_2}\cdots\phi^M_{\alpha_M} \tag{5.1}$$

which obeys the relations

$$\mathbf{TQ}=\Phi_1\mathbf{Q}'+\Phi_2\mathbf{Q}'' \tag{5.2}$$

where Φ_1, Φ_2 are scalars and $\mathbf{Q}'$, $\mathbf{Q}''$ are vectors of the same from as (5.1).

The calculations are closely parallel to those of ref. 25. Rewrite $\mathbf{T}$ as

$$\mathbf{T}^\beta_\alpha=\mathrm{Tr}[\mathbf{L}(\alpha_1,\beta_1)\,\mathbf{L}(\alpha_2,\beta_2)\cdots\mathbf{L}(\alpha_M,\beta_M)] \tag{5.3}$$

where $\mathbf{L}(\alpha,\beta)$ is a two by two matrix

$$L(\alpha,\beta)=\begin{pmatrix} D_{\alpha\beta} & G_{\alpha p} \\ H_{\alpha\beta} & F_{\alpha\beta} \end{pmatrix}$$

whose elements are given by (4.1), (2.12). The product $\mathbf{TQ}$ is a vector which can be written as

$$(\mathbf{TQ})_\alpha=\mathrm{Tr}[\mathbf{K}_1(\alpha_1)\,\mathbf{K}_2(\alpha_2)\cdots\mathbf{K}_M(\alpha_M)] \tag{5.4}$$

$$\mathbf{K}_J(\alpha)=\sum_\beta \mathbf{L}(\alpha,\beta)\phi^J_\beta$$

Note that (5.4) is unaffected if we replace each $\mathbf{K}_J(\alpha)$ by

$$\mathbf{K}^*_J(\alpha)=\mathbf{O}_J^{-1}\mathbf{K}_J(\alpha)\mathbf{O}_{J+1} \tag{5.5}$$

for $J=1,\ldots, M$, provided

$$\mathbf{O}_{M+1}=\mathbf{O}_M \tag{5.6}$$

Choose $\mathbf{O}_J$ of the form

$$\mathbf{O}_J=\frac{1}{m_J}\begin{pmatrix} 1 & -x_J \\ x_J & 1 \end{pmatrix} \tag{5.7}$$

where $m_J=(1+x_J^2)^{1/2}$. The trace in (5.4) will simplify to the sum of two products if we can choose the ϕ_α^J and $\mathbf{O}_J$ so that all $K_J^*(\alpha)$ are upper triangular matrices, i.e., their bottom left elements vanish. This is so if

$$(x_J x_{J+1} G + x_J D - x_{J+1} F - H)\phi^J = 0 \tag{5.8}$$

where D, F, G, and H are defined by (4.1).

Equating the determinant of the coefficients of this linear system to zero, we get

$$x_J^N x_{J+1}^N \det G + x_J^N \det D - x_{J+1}^N \det F - \det H = 0 \tag{5.9}$$

for $J=1,\ldots, M$. Given x_J (5.9) is an Nth-degree equation for x_{J+1}. Thus, if we take x_1 as given, we can construct the entire sequence $x_1,\ldots,x_{M+1}$, having a choice of N alternatives at each stage. To satisfy (5.6), we require that $x_1=x_{M+1}$. Comparing (5.9) with (4.10), one can readily find solutions for which $x_J^N=1$, for $J=1,\ldots, M$. In fact they are the only solutions satisfying (5.6). Hence, we have the N^M solutions of (5.9)

$$x_J=\omega^{\alpha_J}, \qquad \alpha_J=0,\ldots, N-1, \quad J=1,\ldots, M \tag{5.10}$$

where $\omega=\exp(2\pi/N)$, for all N^M choices of $\{\alpha\}=\{\alpha_1,\ldots,\alpha_J\}$. If we use the parametrization (4.11), then $\mathbf{T}, \mathbf{Q}, \phi$ become functions of the rapidities p, q, q'. When necessary we shall write this dependence explicitly. Solving now (5.8) for ϕ^J, we obtain

$$\phi_{\beta_J}^J(p; q, q')=C_1 W_{pq'}(\alpha_J-\beta_J)\, \bar{W}_{pq}(\beta_J-\alpha_{J+1}) \tag{5.11}$$

where W_{pq} and $\bar{W}_{pq}$ are given by (3.3), and C_1 is a normalization factor. Then the vector $\mathbf{Q}$, (5.1), corresponding to the sequence $\{\alpha\}$ in (5.1) has the form

$$\mathbf{Q}_{\{\beta\}}^{\{\alpha\}}(p; q, q')=C_1^M \prod_{J=1}^{M} [W_{pq'}(\alpha_J-\beta_J)\, \bar{W}_{pq}(\beta_J-\alpha_{J+1})] \tag{5.12}$$

Obviously, one can view $\mathbf{Q}_{\{\beta\}}^{\{\alpha\}}$ as the matrix elements of some N^M by N^M

matrix $\mathbf{Q}$, which is nothing but the transfer matrix (3.6a) of the chiral Potts model for the inhomogeneous chain with alternating rapidities q and q'.

We can now calculate the diagonal elements of $\mathbf{K}_J^*$, (5.5), using (5.11) and the relations

$$(\mathbf{K}_J^*)_{00} = \frac{m_J}{m_{J+1}} (D + x_{J+1} G) \phi^J$$
$$(\mathbf{K}_J^*)_{11} = \frac{m_{J+1}}{m_J} (F - x_J G) \phi^J \tag{5.13}$$

Substituting the resulting expression into (5.4) and introducing an index R for $\mathbf{Q}$ to emphasize that $\mathbf{Q}$ is multiplied by $\mathbf{T}$ from the right, we get [for the normalization $W_{pq}(0) = \bar{W}_{pq}(0) = 1$]

$$\mathbf{T}(p; q, q'')\, \mathbf{Q}_R(p; q, q') = \Phi_1(p; q, q')\, \mathbf{Q}_R(R^{N-1}p; q, q') + \Phi_2(p; q, q')\, \mathbf{Q}_R(R^{1-N}p; q, q') \tag{5.14}$$

where R denotes one of the automorphisms of the curve (3.6),[7]

$$p \to Rp$$
$$(a_p, b_p, c_p, d_p) \to (b_p, \omega a_p, d_p, c_p) \tag{5.15}$$

and

$$\Phi_1(p; q, q') = \Phi(p; q, q') \left[\frac{\omega(x_p - x_q \omega^\rho)(t_{q'} - t_p)}{y_{q'} \omega^\rho - x_p} \right]^M \tag{5.16}$$

$$\Phi_2(p; q, q') = \Phi(p; q, q') \left[\frac{(y_p - \omega^{\rho+1} x_{q'})(t_q - \omega t_p)}{\omega^\rho y_q - y_p} \right]^M \tag{5.17}$$

$$\Phi(p; q, q') = [\rho_1 c_p d_p d_q d_{q'} \bar{W}_{pq}(\rho)\, W_{pq'}(-\rho)]^M \tag{5.18}$$

where $\rho = (N-1)/2$, $x_p = a_p/d_p$, $y_p = b_p/c_p$, $t_p = a_p b_p / c_p d_p$. Next, one can find a matrix $\mathbf{Q}_L$ with similar properties. Repeating the calculations and using the fact that

$$\mathbf{T}(R^N p; q, q') = \mathbf{T}(p; q, q') \tag{5.19}$$

we obtain

$$\mathbf{Q}_L(p; q, q')\, \mathbf{T}(p; q, q') = \Phi_1(p; q', q)\, \mathbf{Q}_L(R^{N-1}p; q, q') + \Phi_2(p; q', q)\, \mathbf{Q}_L(R^{1-N}p; q, q') \tag{5.20}$$

where

$$\mathbf{Q}_L(p; q, q') = \mathbf{Q}_R(p; q', q)\hat{\mathbf{P}}^{-1} \tag{5.21}$$

with $\hat{\mathbf{P}}$ given by (3.8). Combining (5.14), (5.20), and (5.21), we have

$$\mathbf{T}(p; q, q')\,\mathbf{Q}_R(p; q, q') = \mathbf{Q}_R(p; q, q')\,\mathbf{T}(p; q', q) \tag{5.22}$$

Setting now $q = q'$, we have

$$\mathbf{Q}_L(p; q, q) = U_{pqq}, \qquad \mathbf{Q}_R(p; q, q) = \hat{U}_{pqq} \tag{5.23}$$

$$[\mathbf{T}(p; q, q), \mathbf{Q}(p; q, q)] = 0 \tag{5.24}$$

where $\mathbf{Q} = \mathbf{Q}_L$ or $\mathbf{Q} = \mathbf{Q}_R$, while U, $\hat{U}$ are defined by (3.6). Moreover, the YBE (2.3) and the star–triangle equation (3.1) imply

$$[\mathbf{T}(p; q, q'), \mathbf{T}(p'; q, q')] = 0 \tag{5.25}$$

$$[\mathbf{Q}(p; q, q)\,\mathbf{Q}(p'; q, q)] = 0 \tag{5.26}$$

Upon uniformizing the curve (3.5), the weights (3.2) and (4.10), and hence the matrices $\mathbf{T}, \mathbf{Q}$ (for the finite length of the chain N) become meromorphic functions of p. Due to commutativity (5.24)–(5.26), $\mathbf{T}$ and $\mathbf{Q}$ have common eigenvectors which are independent of p. Therefore the eigenvalues will be also meromorphic functions of p. Their poles are, of cause, those of the weights.

Consider an action of (5.14) on some eigenvector. Let p^* be any (nontrivial) zero of the corresponding eigenvalue $Q(p; q, q)$; then, from (5.14) we have

$$\frac{Q(R^{N-1}p^*)}{Q(R^{1-N}p^*)} = \frac{\Phi_2(p^*)}{\Phi_1(p^*)} \tag{5.27}$$

which, in principle, fixes of all the zeros of Q and hence contains sufficient data to reconstruct Q. The corresponding eigenvalue of T can then be calculated from (5.14).

The main problem now is to find the suitable uniformization of the curve (3.5). Some progress in this direction was made in ref. 26.

In fact, one can derive additional relations among $\mathbf{T}(p, q, q')$, $U_{p,q,q'}$, and $\hat{U}_{p,q,q'}$. Here we only state the result, emphasizing the most important points of the calculations. The detailed proof given in ref. 19.

The first relation is

$$(L_\alpha^{(2)\beta}(p, q, q'))_{ij} = \Lambda_{ppq'}^{-1} \sum_{\gamma\gamma'\delta} \omega^{\rho(\delta-\gamma)+j\delta-i\gamma} P_{\gamma\gamma'}^{(2)} S_{\gamma'\alpha}^{\beta\delta}(p, R^{N+1}p, q, q') \tag{5.28}$$

where $S_{\alpha\beta}^{\gamma\delta}$ is given by (4.13), $(L_\alpha^{(2)\beta}(p, q, q'))_{ij}$ denotes the matrix elements of the L-operator (4.1), (2.12), $i, j = 0, 1$,

$$P^{(n)}_{\alpha\beta}=\frac{1}{N}\sum_{j=0}^{n-1}\omega^{\rho(n-1-2j)(\alpha-\beta)},\qquad n=1,\ldots,N \tag{5.29}$$

$$P^{(N)}_{\alpha\beta}=\delta_{\alpha\beta} \tag{5.30}$$

$$\Lambda_{pqq'}=\left[\frac{-S_{q',R_p}(x_{q'}-\omega y_p)\,W_{q,R^{N+1}p}(\rho)\,\bar{W}_{q',R^{N+1}p}(\rho)}{(t_{q'}-\omega^2 t_p)(y_q-\omega y_p)\,c_p d_p d_q c_{q'}}\right]^M \tag{5.31}$$

the variables x_p, $y_{p'}$, t_p being defined after Eq. (5.18), and

$$S_{p,q}=N\frac{(t_p^N-t_q^N)(x_p-x_q)(y_p-y_q)}{(t_p-t_q)(x_p^N-x_q^N)(y_p^N-x_q^N)} \tag{5.32}$$

Now consider the fusion procedure for the matrix $\mathbf{T}$, (5.3). For our case this procedure is essentially the same as for the 6v model,[27] because it is determined by the degeneracy point structure of the 6-vertex R-matrix (2.1). We introduce the notations

$$\tau^{(2)}_k=T(R^{\Delta k}p,q,q') \tag{5.33}$$

$$\phi_k=S_{R^{\Delta k+1}p',q} \tag{5.34}$$

$$\mu_k=\Phi_2(p,q,q')\,\Phi_1(R^2p,q,q') \tag{5.35}$$

where Φ_1 and Φ_2 are defined by Eqs. (5.16) and (5.17), and $\Delta=N+1$. Note that $\mu_k(c_pd_pd_qc_{q'})^{-2M}$ is a polynomial in t_p.

The fusion procedure leads to the following relations:

$$\tau^{(k)}_0\tau^{(2)}_{k-1}=\mu_{k-2}\tau^{k-1}_0+\tau^{(k+1)}_0,\qquad k=2,\ldots,N-1 \tag{5.36}$$

where $\tau^{(1)}_k\equiv 1$ and

$$(\tau^{(n)}_k)^{\beta_1\cdots\beta_M}_{\alpha_1\cdots\alpha_M}=\mathrm{Tr}[L^{(n)\beta_1}_{\alpha_1}(R^{\Delta k}p,q,q')\cdots L^{(n)\beta_M}_{\alpha_M}(R^{\Delta k}p,q,q')] \tag{5.37}$$

where $L^{(n)\beta_1}_{\alpha_1}$ are n by n matrices with the elements

$$\begin{aligned}(L^{(n+1)\alpha_n}_{\alpha_0}(p,q,q'))_{ab}&=\binom{n}{a}^{-1}\sum_{\substack{i_1,\ldots,i_n=0\\ j_1,\ldots,j_n=0}}^{1}\delta_{a,i_1+i_2+\cdots+i_n}\,\delta_{b,j_1+j_2+\cdots+j_n}\\&\quad\times\sum_{\alpha_2,\ldots,\alpha_{n-1}=0}^{N-1}\prod_{m=1}^{n}(L^{(2)j_m\beta_m}_{i_m\alpha_m}(R^{(m-1)\Delta}p,q,q'))\end{aligned} \tag{5.38}$$

where $a, b=0,\ldots,n$, and $\binom{n}{a}$ is the binomial coefficient.

Substituting now Eq. (5.28) into (5.38), we obtain

$$\begin{aligned}&\left(\prod_{m=0}^{n-2}\Lambda_m\right)\left(\prod_{m=1}^{n-2}\phi_m\right)^{-1}(L^{(n)\beta}_{\alpha}(p,q,q'))_{a,b}\\&=\frac{1}{N}\sum_{\gamma,\gamma',\delta}\omega^{\rho(n-1)(\delta-\gamma)+b\delta-a\gamma}P^{(n)}_{\gamma\gamma'}S^{\beta\delta}_{\gamma'\alpha}(p,R^{(n-1)\Delta+1}p,q,q')\end{aligned} \tag{5.39}$$

Setting now $n=N$, using (5.30), and remembering the definition (4.13) and (4.6), we get

$$U_{p,q,q'}\hat{U}_{R^N p,q,q'}=\left(\prod_{m=0}^{n-2}\Lambda_m\right)\left(\prod_{m=1}^{n-2}\phi_m\right)^{-1}\tau_1^{(N)} \tag{5.40}$$

Now recall that in our notations $Q_R(p,q,q')$ coincides with $U_{p,q,q'}$. So, using Eq. (5.14) and the recurrence relations (5.36), one can express $\tau^{(N)}$ through U. Substituting the resulting expression into (5.40), we obtain a closed relation which contains only U and $\hat{U}$. In the case $N=3$, $q=q'$, it coincides with the relation presented in ref. 15.

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